Topological insulators and related phases with strong interactions

Ashvin Vishwanath
UC Berkeley

October 17, 2015

0.1 Overview

In these lectures we will mainly be interested in how the concept of topological insulators generalizes when we include interactions. More generally, we discuss the interplay of symmetry and topology. Traditionally, phases of matter were distinguished on the basis of symmetry alone. On the other hand, fractional quantum Hall phases are examples of topological states whose essential character do not require a discussion of symmetry. However, topological insulators are an example of a new phase of matter that combines both symmetry and topology.

To generalize the concept of topological insulators to strongly interacting systems we will need some definitions to limit the set of states we study and hence make progress. Previously, the thinking was that phases like the integer quantum Hall state can occur in free fermion models, and the new physics that interactions bring are fractional Quantum Hall phases. Now we understand that there is some space between these two - there are states that retain the essential physical character of integer Quantum Hall states, but *require* interactions. Partly, the advances occurred by sharply defining what we mean by 'Integer Quantum Hall like' states - by identifying Short Range Entanglement as an essential property.

Throughout we will discuss phases with an energy gap in the bulk - and focus on zero temperature properties. Often we will be interested in new phases of matter, but some of the most striking results will expose phenomena connected to well known phases like topological insulators and superconductors that are obscured by the free particle description.

To whet your appetite, we begin my mentioning three striking theoretical results that emerge on including the effects of interactions. Establishing these will be the goal of these lectures.

- Integer quantum Hall states of electrons have long been believed to be characterized by, well, an integer ($\mathcal{Z}$)- which is the Hall conductance in units of e^2/h. We will see that this is modified in the presence of interactions - *actually the classification is by two integers* $\mathcal{Z} \times Z$, and this family of states retains the essential properties of IQH phases.

- It was believed that if the conducting surface of a 3D topological insulator is made insulating, it must be because the time reversal symmetry is broken, either spontaneously or by application of external fields. It has recently been understood that you can have your cake and eat it too - that*there exist strongly interacting surface phases of a 3D topological insulator, that are insulating but retain time reversal symmetry.* The price you pay is that this state must have fractional excitations at the surface - i.e. ones with fractional charge and fractional (or anyonic) statistics. In fact, it must realize a particularly exotic version of fractional statistics - non-Abelian statistics. The simplest version of this state is closely related to the celebrated Read-Moore Pfaffian state, but with a twist.

- We are used to thinking of the surface states of topological phases (say of 3D phases) as being 'impossible' to realize in a purely 2D system with the same symmetries. Indeed this is generally true - even the new interacting surface phase of a topological insulator cannot be realized in a purely 2D system. However, the set of topological superconductors protected by time reversal symmetry (class DIII) are labeled by an integer ν according to the free fermion classification. Roughly, this counts the number of Majorana cones (which are like 'half' of a Dirac cone) present at the surface. From the free fermion point of view, all these surface states are 'impossible' 2D states. However, with interactions we will show that while $\nu = 1, 2, \ldots 15$ are all indeed impossible in 2D states, *the surface of* $\nu = 16$ *can be realized in a purely 2D but interacting model.* This also means that the integer classification is broken down $\mathcal{Z} \rightarrow \mathcal{Z}_{16}$.

All these are statements about adding interactions to many electron states. But to make progress we will need to take a diversion - and study topological phases of *bosons* or, equivalently, spins. Results there are integrally connected to a deeper understanding of interacting electronic topological insulators and superconductors. Moreover, they might be realized in experiments on ultra cold bosons or frustrated magnetic models. We will discuss some ideas along these lines, but it is fair to say, that conceptual theory is well ahead of experiments and model building in this area. However, the spectacular success of topological insulators in connecting with experiments makes us optimistic. There are even strongly correlated materials proposed to be in this phase. Perhaps some of you will contribute to this direction.

0.2 Lecture Outline

The provisional outline of lectures is as follows:

- Lecture 1 & 2:
  - Basic definitions and properties of short range entangled topological phases.
  - Examples of topological phases of bosons in 1D and 2D from condensing decorated defects.

- Field theory of 2D SRE topological phases of bosons.
- The integer quantum Hall effect revisited - revised classification of interacting Integer quantum Hall states.

- Lecture 3:
  - Topological phase of bosons in 3D. Symmetry fractionalization and topological order. Surface topological order. Walker Wang Models.
  - Interaction effects on topological insulators and superconductors. Surface topological order of electronic topological phases and their stability.

1 Lecture 1

1.1 Quantum Phases of Matter. Short vs. Long Range Entanglement

How do we distinguish different phases of matter? We will be particularly interested in zero temperature states - i.e. the ground state of an interacting bunch of particles. Typically, the phases of matter are only sharply defined in the limit of an infinite number of particles. Then, two states belong to different phases, if there is necessarily a phase transition separating them, where properties change in a singular fashion. For a while, people thought they had figured out how to diagnose this. The answer they believed had to do with symmetry - at the fundamental level, breaking symmetry in different ways lead to different phases. For example - in the Quantum Ising model, with two level systems arranged on a line:

$$H = -J \sum_i \left(\sigma_i^z \sigma_j^z + g \sigma_i^x \right) \tag{1}$$

there is a symmetry or flipping the spin $\sigma^z \to -\sigma^z$ (similarly for the y spin direction). This Z_2 symmetry is spontaneously broken if g is small while it is restored if g is sufficiently large. Thus there are two phases which can be distinguished by the order parameter $\langle \sigma_i^z \rangle$. Symmetry is key to having a sharp distinction - if it is broken by hand, eg. by adding a field along σ^z, then the phase transition can be converted into a crossover. For a while it was thought that all phases of matter (apart from a few well characterized outliers like metals) could be identified by such a procedure.

However, Wegner came up a model which could be shown to have two phases which shared the same symmetry. Today we understand that they differ in their topology - here is the modern avatar of that model, the Kitaev Toric code, which also has two level systems on the vertices of a 2D square lattice[1]. The coupling takes the following form - and includes 4 spin couplings around the plaquettes (see figure 1).

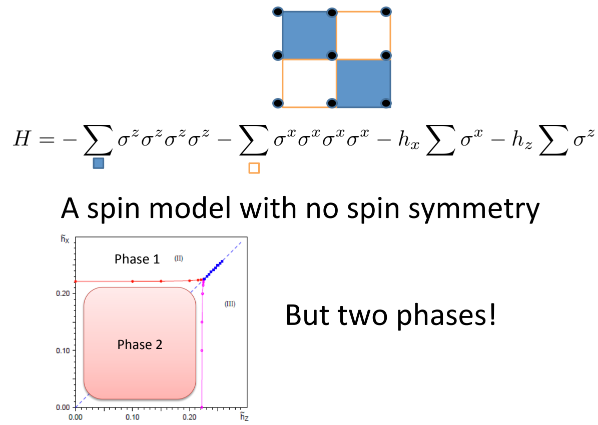

Figure 1: The toric code model with generic perturbations, which has two phases although they have the same symmetry. The Phase 2 is gapped but has long range entanglement - as evidenced by having ground state degeneracy with periodic boundary conditions, and anyone excitations with nontrivial mutual statistics. Phase diagram adapted from Ref. [2]

$$H = -\sum_{black} \sigma_i^z\sigma_j^z\sigma_k^z\sigma_l^z - \sum_{white} \sigma_i^x\sigma_j^x\sigma_k^x\sigma_l^x - h_z \sum_i \sigma_i^z - h_x \sum_i \sigma_i^x \qquad (2)$$

The two phases in this model include a 'trivial' phase which can be thought of as a product state of spins pouting along a certain direction. The other phase does not have any representation as a product state of a site or finite group of sites. It can be thought of as a condensate of closed loops - where the loops are found by linking $\sigma^z = -1$ spins for example. There are two kinds of point excitations in this phase - which violate the individual plaquette terms. One is called a 'charge' and the other the 'vortex'. Despite ultimately being built out of bosons (the spins can equally be thought of as occupying sites with hard core bosons), the excitations have unusual statistics. Taking one around mother leads to a (-1) sign, hence they are mutual semions. This is an example of fractional statistics, in the generalized sense, which includes both exchange of identical particles as well as mutual statistics. This is an indication of long range entanglement (LRE). Another signature is that when the system is defined with periodic boundary conditions (i.e. on a torus), there is a ground state degeneracy. The degenerate states appear identical with respect to any local operator (the degeneracy itself can be understood since the Hamiltonian is a local operator). We define a short range entangled phase as one that does *not* have these properties.

A Short Range Entangled (SRE) state is a gapped phase with a unique ground

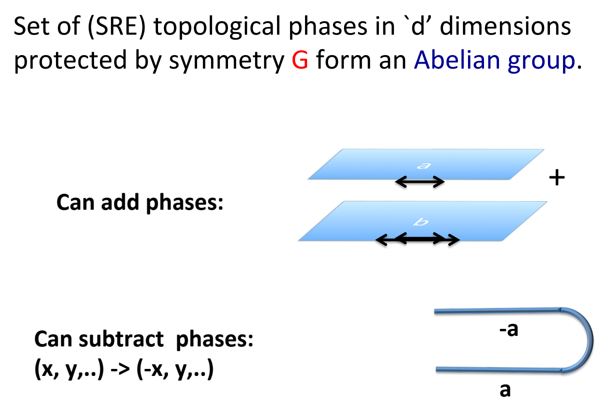

Figure 2: SRE phases that preserve a symmetry must form an Abelian group. This is not true for LRE phases, which typically get more complicated on combining them together.

state on a closed manifold. All excitations (particles with short range interactions) have conventional statistics. For example, if the phase is built of bosons, all excitations are bosonic with trivial mutual statistics.

We also allow for the possibility of a symmetry specified by the group G. We will restrict attention to *internal symmetries* , that is, we do not consider symmetries that change spatial coordinates, like inversion, reflection, translation etc. Common internal symmetries that are encountered in condensed matter physics are charge conservation, various types of spin rotation symmetry, and time reversal symmetry. The advantage of working with internal symmetries is that we can consider disordered systems that respect the symmetry. also the symmetries can be defined at the edge, while for spatial symmetries, one may require a special edge configuration, to preserve symmetry. Some spatial symmetries like inversion are always broken at the edge.

Gapped SRE ground states that preserve their internal symmetries only differ from the trivial phase if they possess edge states. (For 1D systems, the edge states are always gapless excitations, and rigorous statements can be made using matrix product state representation of gapped phases [3, 4, 5, 6].)

The fact that SRE topological phases only differ at the edge, not in the bulk (unlike LRE state), makes them much easier to study. The set of SRE topological phases in a given dimension with symmetry G actually has more structure than just a set. If we add the trivial phase as an 'identity' element, the set of phases actually form an Abelian group. The operations for the group are shown. The addition operation is obvious: take two states and put them side-by-side. But it is not so obvious that a state has an inverse: how is it possible to cancel out edge states? Two copies of a topological insulator cancel one another, because the Dirac points can be coupled by a scattering term that makes a gap. The inverse of a phase is its mirror image (i.e reverse one of the coordinates). To see this, we must show that the state and its mirror image cancel; the argument is illustrated at the bottom of figure 2. In one dimension, for example, take a closed loop of the state, and flatten it. The ends are really part of the bulk of the loop before it was squashed, so they are gapped. Therefore this state has no edge states. Because topological SRE phases are classified by their edge states, it must be the trivial state. Therefore the Hamiltonian describing the inverse of a particular state $H(x, y, z \ldots)$ is, for example $H(-x, y, z \ldots)$. SRE phases protected by a symmetry are termed symmetry protected topological phases (SPTs) [7].

1.2 Examples of SRE Topological phases

Let us give a couple of concrete examples of SRE topological phases of bosons/spins. These are necessarily interacting - unlike free fermions, there is no 'band' picture here.

1.2.1 Haldane phase of S=1 antiferromagnet in d=1

The following simple Hamiltonian actually leads to a gapped phase with SRE, but gapless edge states:

$$H = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

The edge realizes effectively a $S = 1/2$ state, despite the chain being built of $S = 1$ spins. This phenomena has been observed experimentally in some nickel based insulating magnets, e.g. Y_2BaNiO_5. The Ni atoms form S=1 spins, organized into chain like structures that are relatively well isolated from one another .

The symmetry that is crucial to protecting this phase is SO(3) spin rotation symmetry. However, it turns putt hat the full rotation symmetry is not required. It is sufficient to just retain the 180 degree rotations about the x, y and z axes. This symmetry group $\{I, X, Y, Z\}$, contains the identity and the 3 rotation elements. This can be written as $\{I, X\} \times \{1, Y\}$, since $Z = X \times Y$, the combination of two rotations is the third rotation. Mathematically this group is $Z_2 \times Z_2$. We will write down a model with this symmetry (which is not quite reducing the S=1 down to this rotation symmetry), but which has the advantage of being exactly soluble - not just for the ground state but also for all the excited states. This model also has a nice interpretation - of arising from condensing domain walls bound to spin flips.

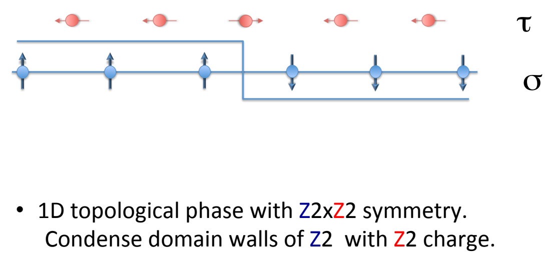

Figure 3: An exactly soluble model of a 1D SRE phase with gapless edge states protected by $Z_2 \times Z_2$ symmetry. Terms in this Hamiltonian encourage binding of domain walls to spin flips. The topological phase emerges on condensing these 'decorated' domain walls.

1.2.2 An exactly soluble topological phase in d=1

Consider a spin model with $Z_2 \times Z_2$ symmetry. There is a Z_2 set of topological phases with this symmetry in d=1, and we will explicitly construct the nontrivial topological phase. We will implement this symmetry by a pair of Ising models (labeled σ and τ) that live on a zig-zag lattice as shown in the Figure. Consider beginning in the ordered state of the σ, but where the τ are disordered and point along a transverse field τ^x. Now, we would like to restore the Z_2 symmetry of the σ spins. We do this by condensing the domain walls of the σ spins. If we directly condense domain walls we get the trivial symmetric state. However, we can choose to condense domain walls with a spin flip of τ attached. We will see that this gives the topological phase [8].

One way to do this is to write down a Hamiltonian that would lead to this binding. Note that the operator $\sigma^z_i \sigma^z_{i+1}$ detects a domain wall. Consider:

$$H = -\sum_i \left(\sigma^z_{2i}\tau^x_{2i+1}\sigma^z_{2i+2} + \tau^z_{2i-1}\sigma^x_{2i}\tau^z_{2i+1}\right) \tag{3}$$

where we have placed the σ (τ) on the even (odd) sites of the lattice. In the absence of a domain wall, we have the usual transverse field term, whose sign changes when a domain wall is encountered. We will show that this is a gapped phase with short range entanglement, but has gapless edge states. In fact this Hamiltonian was previously discussed in the context of generating 'cluster states' [9]

First, consider the system with periodic boundary conditions. We will leave it as an exercise to show that each of the terms in the Hamiltonian 3 commutes with all others. Then, for a system with N sites, we have exactly N terms which can be written as $H = -\sum_i \left(\tilde{\sigma}^x_{2i} + \tilde{\tau}^x_{2i+1}\right)$, where the tilde denote the three spin operators in the Hamiltonian 3. Hence, this simply looks like each site has a modified transverse field, which implies a unique ground state and a gap, in this system with periodic boundary conditions.

Now consider open boundary conditions as shown. Let us focus on the left edge, where the end of the chain implies we lose $\tilde{\sigma}_x$ operator. This will result in a two fold degeneracy as we will show. The first term in the Hamiltonian is now $-\sigma^z_0 \tau^x_1 \sigma^z_2$. We can easily show that the following two operators commute with the Hamiltonian $\Sigma^z = \sigma^z_0$ and $\Sigma^x = \sigma^x_0 \tau^z_1$. However they anti commute with one another. Hence we can show the ground state must be at least two fold degenerate. Say you had a unique ground state of the Hamiltonian, $|\psi\rangle$. This must be an eigenstate of Σ^z, since it commutes with the Hamiltonian. Let us say $\Sigma^z|\psi\rangle = \lambda|\psi\rangle$, where $\lambda = \pm 1$. However, we can find an independent state $|\psi'\rangle = \Sigma^x|\psi\rangle$. This is a degenerate state since $[\Sigma^x, H] = 0$. It is also a distinct state since it has a different eigenvalue $\Sigma^z|\psi\rangle = -\lambda|\psi\rangle$, due to $\Sigma^z\Sigma^x = -\Sigma^x\Sigma^z$. Hence there are at least two ground states ($|\psi\rangle$, $|\psi'\rangle$). They only differ by application of an edge operator, hence this is an edge degeneracy.

Note, it is important we preserve the symmetry - if we add Σ^a to the Hamiltonian we can gap the edge state, but at the expense of also breaking the $Z_2 \times Z_2$ symmetry. Hence this is called a symmetry protected topological phase (SPT). This model has special properties that make it exactly soluble - but adding general perturbations that are local and preserve the symmetry lead to a more generic state. The presence of an energy gap implies that the state is stable against weak perturbation, which means it will remain in the same phase.

Exercises:

1. Verify that the terms in Eqn. 3 commute with one another, in a chain with periodic boundary conditions. With open boundaries, explicitly write out a Hamiltonian and check that the edge operators Σ^a commutes with it.

2. Use the Jordan Wigner procedure to map Eqn. 3 onto fermion operators. Recall, for the 1D quantum Ising model, the transformation is $c^\dagger_j = \sigma^+_j S_j$ where the string operator is $S_j = \prod_{i>j} \sigma^x_i$. Show that the same mapping leads to a topological phase of these non-local fermions. Note however there are some important differences with a topological phase of electrons. Argue that in the latter case there must always be a residual symmetry that cannot be broken by any physical operator, unlike in the fermionized version of the problem above.

Some of this material can be found in a review[10].

2 Lecture 2

Let us give a couple of concrete examples of SRE topological phases of bosons/spins. These are necessarily interacting - unlike free fermions, there is no 'band' picture here. Later, we will see how they combine with the free fermion topological phases, in particular the Integer Quantum Hall effect, to extend the classification from a single integer to a pair of integers.

2.1 SRE phase of bosons in d=2

Let us consider a system of two species of bosons (A and B say these are two species of atoms in an optical lattice), whose numbers are individually conserved. By analogy with the 1D example, we want to find a disordered phase that respects these symmetries, and can be obtained by condensing a vortex combined with a charge. In particular, say we begin in the superfluid state of the 'A' bosons. We want to exit it by condensing vortices (and anti vortices). However, in order to avoid reaching the regular Mott insulator, we will bind a +1 charge of 'B' boson to this vortex (and -1 charge on the anti vortex) before condensing them. We will show that this gives rise to a SRE topological phase. The phase will have gapless edge state protected by symmetry. We will implement this in two ways [11] (i) by a coupled wire construction and (ii) by writing down an effective field theory.

2.1.1 Coupled Wire Construction

Consider an array of wires indexed by i. Species A (B) bosons are placed on the even (odd) wires. Hence they only hop between even or between odd sites. The field theory for a single chain takes the form

$$H = J(\partial_x\phi)^2 + U(\partial_x\theta)^2 \tag{4}$$

where ϕ is the boson phase and θ is defined as

$$\partial_x\theta(x) = 2\pi n(x)$$

where n is the particle density. Using the standard density-phase commutation relation we have $[\partial_x\theta(x), \phi(x')] = -2\pi i\delta(x - x')$. This can be integrated to give:

$$[\theta(x), \phi(x')] = -\pi i \text{Sign}(x - x') \tag{5}$$

where $\text{Sign}(x) = \pm 1$ for $x > 0$ ($x < 0$).

Exercise: Show that this commutation relation is symmetric on interchanging the fields $\theta \leftrightarrow \phi$.

Normally, when the bosons are at commensurate filling one also has a vortex tunneling operators $H_v = -\sum_n \lambda_n \cos(n\theta)$. When these are relevant, a Mott insulator results - for example when $-\cos\theta$ is large, the field is pinned at $\theta = 0$, which implies that the density is uniform as in a Mott state. An excitation is a

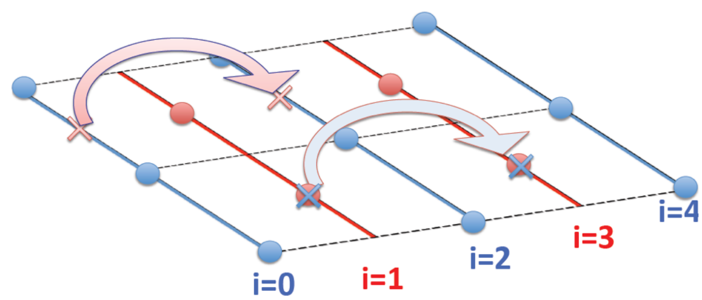

Figure 4: Coupled wire construction of bosonic topological phase. 'A' ('B') bosons, that live on the even (odd) numbered tubes. Vortices of one species are bound to bosons of the opposite species and their tunneling is shown by the arrows. These processes are shown to commute and gap the bulk, but leave behind an topological edge state.

soliton, that is $\theta(x \ll 0) \to 0$ while $\theta(x \gg 0) \to 2\pi$. This costs a finite energy which is the gap to particle excitations in the Mott state.

However, here we will be interested in a different type of vortex condensate, one that binds charge. To this end consider building a 2D system from coupling 1D systems (tubes) together. We will assume that on alternate tubes we have bosons of species the two different species, thus on the even numbered tubes $2i$ we have A and on the odd $2i+1$ we have B. In the absence of interchain coupling w shave the decoupled Luttinger Liquid Hamiltonian:

$$H_0 = K \sum_i \left[(\partial_x \theta_i)^2 + (\partial_x \phi_i)^2 \right] \tag{6}$$

and

$$[\theta_j(x), \phi_k(x')] = -i\pi \delta_{j,k} \delta(x - x') \tag{7}$$

We ignore the effect of conventional vortices. Let us instead attempt to condense vortices of the 'A' bosons with B charge and vice versa. We will do this by allowing the composite objects to 'hop', which will lower their energy and make them eventually 'Bose condense'. Of course, since this is a vortex condensate we are led to an insulator as elaborated previously. We will see that this is an exotic insulator.

Note, the vortices of the 'A' (blue) bosons naturally live in the centers of plaquettes of the sites available to 'A'. This happens to be on the 'B' tubes as shown by the blue cross in the figure. Hence, if we hop a vortex from one of the blue crosses to the adjacent one, this can be represented as a space-time event on tube $i = 2$, represented as $e^{i\theta_2}$. However, we also simultaneously want to hop bosons 'B' and we have arranged for their lattices sites to coincide with the location of the 'B' vortices. This combined process is then written as: $e^{i(-\phi_1+\phi_3+\theta_2)}$. The reverse process binds an anti-vortex to a 'hole': $e^{i(+\phi_1-\phi_3-\theta_2)}$ and taken to gather this leads to the set of terms:

$$H_{int} = -\lambda \sum_i \cos(\phi_{i-1} - \theta_i - \phi_{i+1}) \tag{8}$$

First we note that these terms all commute with one another, for example, if we denote $\tilde{\theta}_i = \theta_i + (\phi_{i+1} - \phi_{i-1})$, then any two terms commute:

$$\left[\tilde{\theta}_i(x),\, \tilde{\theta}_j\Big(x')\right] = 0$$

Exercise: Calculate the commutator of a pair of fields $\Phi_{l,m}$ and $\Phi_{l',m'}$ where $\Phi_{l,m} = \sum_i (l_i\phi_i + m_i\theta_i)$. Use this to prove the result above.

Thus all of these can be simultaneously satisfied. If we have periodic boundary conditions, then there is a unique ground state rather like pinning the θ fields gives a unique Mott insulating state. The same count of variables leads to a unique state in this case. However, interesting edge states appear if we have an open slab as in the Figure 4. Let us focus on the left edge. Clearly we are missing a cosine pinning field for $\tilde{\theta}_0$. This means that the first non vanishing cosine terms are $\tilde{\theta}_1 = -\phi_0 + \theta_1 + \phi_2$ and $\tilde{\theta}_2 = -\phi_1 + \theta_2 + \phi_3$. An field that commutes with both these is $\Phi = \phi_0$. However, there is also a conjugate field we require to define the edge dynamics. In an isolated chain this would have been θ_0 However, here this does not commute with one of the first cosine. It can be rectified by adding $\Theta = \theta_0 + \phi_1$, which commute with all the cosines, and has the standard Luttinger liquid commutator with Φ: $[\Theta(x),\, \Phi(x')] = -i\pi\delta(x - x')$. Hence we have a gapless edge mode - which is described by the usual Luttinger liquid theory. However, there is an important difference between this Luttinger liquid at the edge and one that can be realized in purely 1D. It is impossible to gap this edge without breaking one of the symmetries, which happens to be number conservation of the two boson species. This is an internal symmetry - and in a purely 1D system it is always possible to find a gapped state that preserves all internal symmetries - we just combine degrees of freedom till they transform in a trivial way under the symmetry and condense them. However, this is not possible at the edge - one cannot condense Φ since it is charged under the U(1) symmetry that protects 'A' particle conservation, and similarly we cannot introduce a cosine of the Θ field since it is charged under the other U(1). This is an indication that it is a topological phases - we will see this also implies a quantized Hall conductance.

2.1.2 Effective Field Theory

Let us write down an effective theory to describe a fluid built out of boson-vortex composites[11]. The 'A' particles acquire a phase of 2π on circling vortices, hence the effect go vortices can be modeled by a vector potential whose curl is centered at the vortex locations: $\partial_x a_y - \partial_y a_x = 2\pi \sum_j n_j^v \delta(r - r_j^v)$, where $(n_j^v,\, r_j^v)$ represent the strength and location of the vortices. This vector potential will couple minimally to the current $\mathcal{L} = \vec{j}_A \cdot \vec{a}$, where the vectors are two-vectors. A rewriting o this formalism to include motion of vortices results in the following generalization to the three current $j^\mu = (\rho,\, j_x,\, j_y)$, and three gauge potential $a_\mu = (a_0,\, a_x,\, a_y)$. Also, since we assume the vortices are bound to the bosons 'B' we can rewrite the equation for a as:

$$\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^B = 2\pi j_B^\mu \tag{9}$$

where we have introduced a superscript 'B' for the vector potential. At the same time we can utilize the continuity equation for the current j_A, $\partial_\mu j_A^\mu = 0$ to write:

$$\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^A = 2\pi j_A^\mu \tag{10}$$

In order to keep track of the charge density of 'A' and 'B' bosons, it is useful to introduce external vector potentials $A^{(A,\,B)}$ that couple to the currents of the 'A' and 'B' bosons. This leads to our final topological Lagrangian:

$$\mathcal{L}_{topo} = \frac{\epsilon^{\mu\nu\lambda}}{2\pi}\left(a_\mu^A \partial_\nu a_\lambda^B + A_\mu^A \partial_\nu a_\lambda^A + A_\mu^B \partial_\nu a_\lambda^B\right) \tag{11}$$

$$\mathcal{Z}_{topo}[A^A, A^B] = e^{iS_{topo}} = \int \mathcal{D}a^A \mathcal{D}a^B e^{i\int dxdydt\, \mathcal{L}_{topo}} \tag{12}$$

First, we would like to establish that this describes a phase with short range entanglement. Note, the mutual phases involved are all 2π implying the absence of fractional statistics. One can also compute the ground state degeneracy on the torus - this turns out to be directly computable from this theory - if we write

$$\mathcal{L} = \frac{K_{IJ}}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu^I \partial_\nu a_\lambda^J$$

, the ground state degeneracy is $|detK|$. In this case $K = \sigma^x$, and there is a unique ground state.

Given that it is a SRE phase, we can deduce two important consequences from this theory- the first is regarding edge states, which can be shown to be equivalent to that derived before, and the quantized Hall conductivity. The latter is obtained by integrating out the a fields to obtain an action purely in terms of the external probe fields A. The current is then defined as $j_A = \frac{\delta S}{\delta A^A}$ where $Z = e^{iS}$. A gaussian integration of Eqn. 12 yields:

$$S_{topo} = -\int dxdydt\, \frac{\epsilon^{\mu\nu\lambda}}{2\pi} A_\mu^A \partial_\nu A_\lambda^B \tag{13}$$

thus we have:

$$j_A^\mu = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda^B \quad (14)$$

If we consider the spatial components of this equation we find: $j_A^x = \frac{1}{2\pi}E_B^y$, where E_B is the electric field applied on species 'B'. Thus we have a crossed response Hall conductivity $\sigma_{xy}^{AB} = \frac{1}{2\pi}$, which , replacing charge Q_a for the bosons and $\hbar$ gives $\sigma_{xy}^{AB} = \frac{Q_A Q_B}{h}$.

We would like to apply these insights to electronic systems, where one may combine pairs of electrons to form Cooper pairs with charge $Q = 2e$. However, in that case there is a single conservation law. Topological phases with a single $U(1)$ can be described by the field theory above 12 , if we assume that the two species of bosons can tunnel into one another and collapse the combined $U(1) \times U(1)$ symmetry into a single common $U(1)$. This amounts to replacing the pair of external vector potentials by a single one and the resting topological response theory is:

$$S_{topo} = -\int dxdydt\, \frac{\epsilon^{\mu\nu\lambda}}{2\pi}A_\mu\partial_\nu A_\lambda \quad (15)$$

Now, differentiating with respect to the vector potential we get *two* contributions and hence: $j^\mu = \frac{2}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$ which implies a Hall conductivity, in units of the boson charge $\sigma_{xy} = 2\frac{Q^2}{h}$. This is the Bosonic Integer Quantum Hall (**BIQH**) phase. Somewhat surprisingly, its Hall conductance is always an *even* integer. Potential realization of this phase in bilayer systems of bosons in the lowest Landau level with net filling $\nu = 2$ have been discussed in recent numerical work [1].

Note we have assumed commensurate filling to admit an insulator. Also, these models are not exactly soluble in the same way that the previous models were - for other approaches to construct models in this phase see [12] .

2.1.3 Implications for Interacting Quantum Hall State of Electrons:

It is well known that free fermion IQH states have a quantized Hall conductance $\sigma_{xy} = n\frac{e^2}{h}$. At the same time, they have a quantized thermal hall effect $\frac{\kappa_{xy}}{T} = c\frac{\pi^2 k_B^2}{3h}$ where $c = n$. The latter simply counts the difference between the number of right moving and left moving edge states. This equality is an expression of the Wiedemann Franz law that related thermal and electrical conductivity for weakly interacting electrons. This leads to the familiar integer classification of IQH $\mathcal{Z}$. How is this modified in the presence of interactions? We will continue to assume short range entanglement - so that fractional quantum Hall states are excluded from our discussion. It has long been known that n must remain an integer if charge is to remain unfractionalized. However, the equality $n = c$ can be modified. In fact, if we assume the electrons can combine into Cooper pairs which form the BIQH state, the latter has Hall conductance $\sigma_{xy} = 8\frac{e^2}{h}$

[1] See arXiv:1305.0298, arXiv:1304.5716, arXiv:1304.7553

but $\kappa_{xy} = 0$. Thus we can have $n - c = 8m$. Indeed this implies that the classification of interacting quantum Hall states of electrons with SRE is $\mathcal{Z} \times Z$ at least. Note, this also predicts a phase where $n = 0$ but $c = 8$. This can be achieved by combining an $n = 8$ free fermion quantum Hall state with a BIQH state of Cooper pairs to cancel the electrical Hall conductance. The remaining thermal Hall conductance is $c = 8$. It can be shown that a π flux inserted in this state has trivial statistics and can be condensed - which implies that all electrons are confined into bosonic particles without disturbing the topological response of this phase[13]. Alternately, one can show that neutral bosons with short range interaction can lead to a topological phase with chiral edge states, if they appear in multiples of eight. Indeed one can write down a multi component chern simons theory to describe this topological phase of neutral bosons, in terms of a K matrix as described in detail below [11].

A phase without topological order is characterized by a symmetric K matrix with $|\det \mathbf{K}| = 1$. A chiral state in $2+1$-D requires the signature (n_+, n_-) of its K matrix to satisfy that $n_+ \neq n_-$. We therefore seek a K matrix with the following properties (1) $|\det \mathbf{K}| = 1$ (2) the diagonal elements $K_{I,I}$ are all even integers so that all excitations are bosons and (3) a maximally chiral phases, where all the edge states propagate in a single direction. Then, all eigenvalues of K must have the same sign (say positive), so $\mathbf{K}$ is a positive definite symmetric unimodular matrix.

It is helpful to map the problem of finding such a $\mathbf{K}$ to the following crystallographic problem. Diagonalizing $\mathbf{K}$ and multiplying each normalized eigenvector by the square root of its eigenvalue one obtains a set of primitive lattice vectors $\mathbf{e}_I$ such that $K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$. The inner product of a pair of vectors $l_I \mathbf{e}_I$ and $l'_I \mathbf{e}_I$ are given by $l'_I K_{IJ} l_J$, while the volume of the unit cell is given by $[\mathrm{Det} K]^{1/2}$. The latter can be seen by writing the components of the vectors as a square matrix: $[\mathbf{k}]_{aI} = [e_I]_a$. Then $\mathrm{Det} k$ is the volume of the unit cell. However, $K_{IJ} = \sum_a k_{aI} k_{aJ} = (\mathbf{k^T k})_{IJ}$. Thus $\mathrm{Det}\mathbf{K} = [\mathrm{Det}\mathbf{k}]^2$.

Thus, for a phase without topological order, we require the volume of the lattice unit cell to be unity $[\mathrm{Det} k] = 1$ (unimodular lattice). Furthermore, for a bosonic state, we need that all lattice vectors have even length $l_I K_{IJ} l_J =$ even integer, since the K matrix has even diagonal entries (even lattice). It is known that the minimum dimension this can occur in is eight. In fact, the root lattice of the exceptional Lie group E_8 is the smallest dimensional unimodular, even lattice[2]. Such lattices only occur in dimensions that are a multiple of 8. See Ref [14] for more details.

[2] See wikipedia entry for E_8 root lattice (Gosset lattice)

A specific form of the K matrix is:

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \tag{16}$$

This matrix has unit determinant and all eigenvalues are positive. It defines a topological phase of bosons without topological order, with eight chiral bosons at the edge. We will call this the E_8 state since it is related to the E_8 group.

3 Lecture 3

3.1 SPT phases of bosons in d=3

We would like to understand how to describe the surface states of a 3D SRE topological phase of bosons. While for free fermions, one has Dirac or Majorana surface cones, the bosonic analog is less clear, particularly since the surface is 2D and one does not have access to bosonization and other powerful tools available for the previous problem of 1D edges.

Based on our previous experience with 1D edges, we will directly consider the surface and ask how symmetry can act in an anomalous way, to produce a topological surface[15]. The simplest example is to consider a system with $U(1)$ and $\mathcal{T}$ symmetry, where the $U(1)$ may be considered as a conserved spin component (S_z) rather than charge. This corresponds to $U(1) \times \mathcal{T}$. One option is to break the symmetry at the surface - this is a valid surface state even for a topological bulk. Say we break the $U(1)$ to get an ordered surface (which we will call a 'superfluid' since it breaks a U(1) symmetry). To restore this symmetry we would like to proliferate vortices, that can revert us to the fully symmetric state. However, for the surface of a topological bulk, there should be an obstruction to proliferating vortices. The rolling is a potential mechanism - note the vortices here preserve time reversal symmetry, that is, a vortex is mapped to a vortex under $\mathcal{T}$. This follows from the fact that our phase degree of freedom transforms like magnetic order with $\phi \to \phi + \pi$ under time reversal, so that $e^{i\phi} \to -e^{i\phi}$. The vorticity, which is defined via $\nabla \times \nabla\phi$ is invariant under this operation. Hence we can ask - how does a vortex transform under $\mathcal{T}$?

There are two physically distinct options, whether the vortex transforms as a regular, or a projective representation. In the former case there is no obstruction to condensing vortices and restoring the symmetric phase - hence this cannot represent a topological surface state. However, the vortices can also transform as a projective representation since they are nonlocal objects. On a closed surface one must make a vortex-antivortex pair. Taken together these must transform as $\mathcal{T}^2 = +1$. However, individually they can transform as $\mathcal{T}^2 = -1$, i.e. the vortex is a Kramers doublet. Denote ψ_σ as the two component vortex field $\sigma =\uparrow, \downarrow$, which transforms as $\psi_\uparrow \to \psi_\downarrow$, $\psi_\downarrow \to -\psi_\uparrow$ under time reversal . The effective Lagrangian is:

$$\mathcal{L} = |(\partial_\mu - ia_\mu)\psi_\sigma|^2 + (\partial_\mu a_\nu - \partial_\nu a_\mu)^2 + m|\psi_\sigma|^2 + \dots \tag{17}$$

where the gauge field is determined by the bosons three current

$$\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda = 2\pi \jmath^\mu \tag{18}$$

, which includes the boson charge density j^0 and current $j^{1,2}$. The vortex-gauge field coupling is intuitively rationalized from the fact that a vortex moving around a boson acquires a 2π phase. Hence, the gauge potential a that implements this satisfies: $\partial_x a_y - \partial_y a_x = 2\pi j^0$. This is one component of the equation 18 above, the other components follow from the continuity equation $\partial_\mu j^\mu = 0$.

In this dual language, when the vortices are gapped the $U(1)$ symmetry is broken, while if they are condensed the $U(1)$ symmetry is restored. The key difference between a single component vortex field, and the Kramers doublet vortex, is that in the latter case the vortex condensate always breaks time reversal symmetry. This can be seen by considering the operator $\psi^\dagger_\sigma \sigma^a_{\sigma\sigma'}\psi_\sigma = n^a$, where σ^a are Pauli matrices. Since it is a product of a vortex-antivortex pair, it is a local operator unlike an operator that insets a vortex. Ina vortex condensate this operator will acquire a nonzero expectation value. Under time reversal it is readily seen $n^a \to -n^a$, indicating that time reversal symmetry is broken. Thus, the $U(1)$ symmetry is restored at the expense of breaking $\mathcal{T}$. This is a candidate for a topological surface state.

Exercise Establish this by introducing an external 'probe' electromagnet field that couples to the bosons $\mathcal{L}_{int} = j^\mu A_\mu = A_\mu \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda / 2\pi$ and integrate out the other fields to obtain an effective action in terms of A. Consider doing this in two limits $m > 0$ ($m < 0$), where vortices are gapped (condensed) Show that when the vortices are gapped, the effective Lagrangian is $\mathcal{L}_{eff} \sim A^2_\perp$, where $A_\perp$ is the transverse part, and this represents a U(1) broken phase ('superfluid'). On the other hand, when the vortices are condensed show that $\mathcal{L}_{eff} \sim (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$ (an 'insulator').

We mention two other possible surface states that this theory.

The first is the critical point $m = 0$, where symmetries are unbroken, but the surface is gapless. This is the bosonic analog of the gapless Dirac cone of fermionic topological insulators. However, since bosons are either gapped or condensed, this requires tuning a parameter to realize. This field theory (the non compact CP^1 model [16]) appeared before in the theory of 'deconfined quantum critical points', describing a direct transition between Neel and Valence bond solid order in spin models on the square lattice [17, 18]. However, there the vortices transformed projectively under spatial symmetries - such as translation and rotation. Here, an internal symmetry (time reversal) is involved - which can only occur on the surface of a 3D topological phase.

3.1.1 Surface Topological Order of 3D Bosonic SRE Phases

The second possibility is to consider condensing a pair of vortices $\Phi = \epsilon_{\sigma\sigma'}\psi_\sigma(r)\psi'_\sigma(r')$, which is a Kramers singlet. This leads to a restoration of the $U(1)$ symmetry (insulator), while preserving $\mathcal{T}$. However, this is an 'exotic' insulator with topological order (excitations that fractional statistics). Note however, the bulk 3D state is still SRE, and the exotic excitations are confined to the surface. It is readily shown that the topological order is the same as that in the toric code. Note, to show this we need to identify an e and m particle which are bosons, but with π mutual statistics. The m particle is just the unpaired vortex, which remains as a gapped excitation in this phase. Additionally, we can discuss defects in the 2-vortex condensate. These are nothing but particles - however, the 2-condensate allows for a fractional particle. To see this consider the the effective 2-vortex theory $\mathcal{L}_{2v} = |(\partial_\mu - 2ia_\mu)\Phi|^2 + (\partial_\mu a_\nu - \partial_\nu a_\mu)^2 + m_2|\Phi|^2 + \ldots$, which can be obtained from (17) by considering an interaction that pairs vortices and ignoring the gapped single vortices. In the 2-vortex condensate one can consider vortices - which are obtained from the flux quantization condition $2(\partial_x a_y - \partial_y a_x) = 2\pi$, but since the flux is related to particle density, this implies a particle with charge 1/2 that of the fundamental bosons. Clearly, taking a half charge around a vortex leads to π phase. Hence this is the m particle.

This surface topological order provides a powerful way to characterize a 3D topological phase [15]. The surfaces of SRE topological phases should be distinct from states that can be realized purely in the lower dimension. The way this works with surface topological order is that although the topological order itself can be realized in 2D, the way the excitations transform under symmetry cannot be realized in a purely 2D setup. For example here the m particle is a Kramers doublet while the e particle carries half charge of the boson (and may or may not be a Kramers doublet).

While in this case it is not immediately apparent that this is forbidden in 2D, we can give another example that arises where this is obvious. Consider the situation where both e and m particles carry half charge - this is one of the surface topological orders associated with U(1) charge and T symmetry. . We can show that this state cannot be $\mathcal{T}$ symmetric if realized in 2D, where it can be described by a K matrix CS theory:

$$\mathcal{L}_{CS} = \frac{2}{2\pi} a_1 \cdot \nabla \times a_2 - \frac{\nabla \times A}{2\pi} \cdot (a_1 + a_2) \tag{19}$$

coupling to the external A ensures that we can keep track of the charge. Note, $K = 2\sigma_x$ ensures we have toric code type Z_2 topological order (|Det K|=4). Now, integrating out a, we obtain $\mathcal{L}_{eff} = -\frac{1}{4\pi} A \cdot \nabla \times A$. This implies that if this state is realized in 2D it will have a non vanishing Hall conductance, $\sigma_{xy} = Q^2/h$ contradicting the fact that it is $\mathcal{T}$ symmetric. However it can be realized retaining $\mathcal{T}$ symmetry on the surface of a topological phase.

The simplest way to argue this is the following construction coupled layer construction[19], analogous to the 1D and 2D cases that we discussed before. Consider layers of 2D toric code models where just the e particle carries half

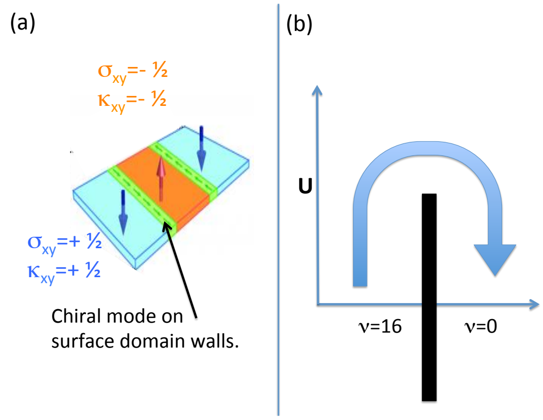

Figure 5: (a) Magnetic domains on the surface of a 3D free electron Topological insulator. The resulting insulating surface has $\sigma_{xy} = \pm 1/2(e^2/h)$ and $\kappa_{xy} = \pm 1/2$, since the domain wall carries a single chiral edge mode. (b) Interaction effects on the surface of a 3D Topological superconductor are irrelevant for weak interactions, given the linear dispersion of the Majorana cone surface states. However strong interactions 'U' can connect two surfaces without a phase transition - which implies the bulk phases are equivalent in the presence of interactions. Establishing this requires nonperturbative techniques.

charge. Now, in parallel to the constructions in lower dimensions, we consider a set of 3 layers, and form a bound state of $e_i m_{i+1} e_{i+2}$. This is a boson, which commutes with other triplets. For example $e_0 m_1 e_2$ and $e_1 m_2 e_3$ are mutual bosons. Also, it has integer charge and can be neutralized by a physical bosons. Hence condensing these triplets leads to a SRE 3D state, with all symmetries. However, it leaves behind an edge state - eg. e_0 is not confined. Similarly $m_0 e_1$ also commutes with the condensate. This is the new m particle of the toric code topological order, which is confined to the top layer. Note that it carries half charge, which is precisely what we wanted to construct. Here time reversal is explicitly preserved. By realizing this state on the surface of a 3D system ensures we never have to declare the edge physics (which would break time reversal symmetry).

This state corresponds to the surface of a 3D bosonic topological insulator (3D BTI), and models a surface, which is 'half' the 2D bosonic Integer Quantum Hall phase which has $\sigma_{xy} = Q^2/h$. Note, one can draw the following analogy to the free fermion topological insulator. A time reversal symmetry breaking perturbation can render the surface of the 3D TIU insulating. However, a domain wall between two opposite T-breaking domain on the surface necessarily has a single chiral mode along it (see Figure 5). Therefore the difference in Hall conductivity between the two domains is $\Delta\sigma_{xy} = 1(e^2/h)$. Also $\Delta\kappa_{xy} = 1$. By time reversal symmetry the two domains should have opposite Hall conductivities - hence we are forced to assign $\sigma_{xy} = +\frac{1}{2}\frac{e^2}{h}$ and $\kappa_{xy} = +\frac{1}{2}$ and the time reversed version to the other domain. Since a purely 2D free fermion system cannot have fractional Hall conductivities, it is not possible to screen this with a 2D layer. In a similar way we can build a 3D topological phase from the 2D integer Quantum Hall state of bosons, by including time reversal symmetry. This is the 3D BTI, whose surface state is described above.

In a very similar fashion one can model a state with a surface that is 'half' the chiral E_8 state, but time reversal symmetric when realized in 3D. This is the 3D bosonic topological superconductor (3D BTSc), and although is a symmetry protected topological phase, is not captured by the 'cohomology' approach [7, 20]. The surface topological order is the fermionic variant of the toric code - it has three nontrivial particles that have mutual π statistics, like the toric code, but all three particles have fermionic statistics. At first sight it might appear that this state is time reversal symmetric - but in fact it must carry chiral edge modes if realized in 2D. An explicit Chern Simons representation of this state is provided through the K matrix:

$$K = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix} \tag{20}$$

The eigenvalues of this matrix are all the same sign, implying that there are 4 chiral modes if this state is realized in 2D, and hence always breaks $\mathcal{T}$ symmetry. However it may be realized on the surface of a 3D topological state with $\mathcal{T}$. Again this may be obtained via a coupled layer construction. A different approach [21] to realizing this phase is via an exactly soluble model, based on the following observation. It is well known in the context of the 2D Fractional Quantum Hall effect, that ground state wavefunctions can be related to correlation functions of the edge conformal field theory. The two coordinates of particles in the wave function are traded for a single spatial coordinate at the edge, and time. Can a similar approach be taken for 3D topological phases? While the obvious generalization is to relate the wave function written in terms of particle coordinates, a useful generalization is obtained by representing the wave function in terms of loops. Now, the amplitude of a particular loop configuration $\mathcal{C}$ in 3D space, $\Psi(\mathcal{C})$, is related to the space-time amplitude for a process in which the loops are imagined as world lines of particles in the surface topological order. Hence the loops come in different 'colors' corresponding to the nontrivial particles in the theory, and rules concerning how they fuse together etc. are determined by the topological data of the surface theory. For example, in the case of the 3-fermion topological order, the amplitude is

$$\Psi(\mathcal{C}) = \int Da\, e^{i\oint_{\mathcal{C}} j^\mu_I a^I_\mu} e^{iK_{IJ}\int \epsilon^{\mu\nu\lambda} a^I_\mu \partial_\nu a^J_\lambda} \tag{21}$$

where the j define the loop structure. This can be converted into a exactly soluble model (Walker Wang model) on the cubic lattice .

3.1.2 Surface topological order of fermionic topological insulators and superconductors.

The well known fermionic Z_2 topological insulator is usually associated with a single dirac cone surface state. Breaking time reversal symmetry at the surface (eg. by introducing magnetic moments that order) can open a surface gap and render it insulating. However, this is not the only way to obtain an insulating surface - one can preserve all symmetries and obtain an insulator with topological order as for the bosonic SRE phases. Note, by the same logic as for the 3D BTI and 3D BTSc, the topological order is such that when realized in purely 3D it breaks $\mathcal{T}$ symmetry, and has $\sigma_{xy} = \frac{1}{2}e^2/h$ and $\kappa_{xy} = \frac{1}{2}$. That is - it is a candidate for a fractional Quantum Hall effect of electrons in a half filled Landau level. The most famous such candidate is the Moore-Read Pfaffian state, which may be thought of as Ising $\times U(1)_8$. More physically, one can imagine beginning with a superconductor of electrons, in a $p_x + ip_y$ state, where the Cooper pairs are effectively at $\nu_{Cooper} = 1/8$ filling [3] When the Cooper pairs form a bosonic Laughlin state, the Moore-Read state results. Unfortunately, while this state has the right σ_{xy} it has $\kappa_{xy} = 3/2$. Moreover, a quick glance at the topological spins of the quasiparticles reveals that it cannot be made time reversal symmetric even on the surface of a 3D topological insulator. Fortunately a simple variant is much more promising [22, 23]- one considers $p_x - ip_y$ superconductor in conjunction with the same Cooper pair Laughlin state, i.e. Ising$^* \times U(1)_8$. This state, dubbed the T-Pfaffian[22] can be made time reversal symmetric on the surface of a 3D TI, but of course breaks it in 2D since it has a finite Hall conductance. A different but equivalent solution [24, 25] features the Moore-Read state in conjunction with a neutral anti-semion theory $U(1)_{-2}$. The surface topological order helps to understand how this classification is augmented in the presence of strong interactions wherein the electrons may pair to form bosons that exhibit a topological phase. The electrons could form Cooper pairs that then go into a 3D BTI phase. Or, the electrons could combine into neutral bosons that then enter a 3D BTSc phase. Both of these extend the original Z_2 Classification by an additional factor of Z_2. Wang, Potter and Senthil [26] showed that this exhausts the set of 3D topological phases of interacting electrons with charge conservation and $\mathcal{T}$ symmetry.

The topological superconductors in 3D are protected by $\mathcal{T}$, and, for the physical case of $\mathcal{T}^2 = -1$ when acting on fermions, gives rise to an integer set of topological phases. One may imagine that combining pairs of electrons into neutral bosons, one can augment this classification by Z_2, by including the 3D BTSc in this list. However, it turns out that this phase is already present in the free fermion classification and corresponds to number $\nu = 8$ of the $\mathcal{Z}$ classification [27]. Hence, there is no new phase. On the other hand, since this phase has a Z_2 classification, this implies that two copies of $\nu = 8$, i.e. $\nu = 16$ is trivial. Therefore the interacting topological superconductor classification

[3] $\nu_{Cooper} = \frac{1}{4}\nu_{electron}$, since there are half as many Cooper pairs as electrons, and the magnetic field measured in units of the new flux quantum $h/2e$ is twice as large.

Symmetries	Free Fermions	Interacting Bosons	Interacting Fermions
U(1) (charge) and $\mathcal{T}$ Topological Insulator	Z_2 Class AII	Z_2^3	Z_2^3
$\mathcal{T}$ $\mathcal{T}^2 = (-1)^{N_F}$ TSc	Z Class DIII	Z_2^2	Z_{16}

Table 1: Topological phases in 3D with short range entanglement. The physically most relevant symmetries, corresponding to the topological insulator and superconductor.

is reduced from the free fermion one $\mathcal{Z} \to \mathcal{Z}_{16}$[27, 28]. This observation is interesting since it represents a non-perturbative result in 2+1 dimensions and has been verified by other means [29, 30, 31, 32]. The surface of a topological superconductor with $\nu = 1$ has a Majorana cone with low energy dispersion:

$$H = -i\chi^T \left(\sigma^x \partial_x + \sigma^z \partial_y\right) \chi \tag{22}$$

where $\chi^T = (\chi_1, \chi_2)$. The surface with index ν then has ν flavors with the dispersion above. It is readily verified that weak interactions at the surface are irrelevant. From Eqn. 22, requiring that the action corresponding to the kinetic term is dimensionless (and both time and space have same dimensions $[t] = [x] \sim L$ the scaling dimension of the χ fields are $[\chi] \sim \frac{1}{L}$. The interaction term, written schematically as $\mathcal{S}_{int} \sim \int d^2dt\, (\chi_a^T \sigma_y \chi_a)(\chi_b^T \sigma_y \chi_b)$ then has scaling dimension $[\mathcal{S}_{int}] \sim \frac{1}{L}$ which means that it is irrelevant at long scales. Therefore, the way $\nu = 16$ is connected to $\nu = 0$ is via strong interactions as shown in the figure 5b. Establishing this therefore requires a nonperturbative analysis. While bosonization provides such a tool for a 1+1D edge, establishing this for 2+1D requires new non perturbative tools - such as working with the surface topological order or a dual vortex theory/monopoles [33]. A nice review of these recent developments is in Ref. [34].

References

[1] A. Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals Phys.*, 303:2, 2003.

[2] I. S. Tupitsyn, A. Kitaev, N. V. Prokof'ev, and P. C. E. Stamp. Topological multicritical point in the phase diagram of the toric code model and three-dimensional lattice gauge higgs model. *Phys. Rev. B*, 82:085114, Aug 2010.

[3] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen. Classification of gapped symmetric phases in one-dimensional spin systems. *Phys. Rev. B*, 83:035107, Jan 2011.

[4] Lukasz Fidkowski and Alexei Kitaev. Topological phases of fermions in one dimension. *Phys. Rev. B*, 83:075103, Feb 2011.

[5] Ari M. Turner, Frank Pollmann, and Erez Berg. Topological phases of one-dimensional fermions: An entanglement point of view. *Phys. Rev. B*, 83:075102, Feb 2011.

[6] Norbert Schuch, David P?rez-Garc?a, and Ignacio Cirac. Classifying quantum phases using matrix product states and projected entangled pair states. *Phys. Rev. B*, 84(16):165139–, October 2011.

[7] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen. Symmetry-protected topological orders in interacting bosonic systems. *Science*, 338(6114):1604–1606, December 2012.

[8] Xie Chen, Yuan-Ming Lu, and Ashvin Vishwanath. Symmetry protected topological phases from decorated domain walls. *Nature Commun.*, 5:3507, 2014.

[9] Pietro Smacchia, Luigi Amico, Paolo Facchi, Rosario Fazio, Giuseppe Florio, Saverio Pascazio, and Vlatko Vedral. Statistical mechanics of the cluster ising model. *Phys. Rev. A*, 84:022304, Aug 2011.

[10] Ari M. Turner and Ashvin Vishwanath. Chapter 11 - beyond band insulators: Topology of semimetals and interacting phases. In Marcel Franz and Laurens Molenkamp, editors, *Topological Insulators*, volume Volume 6, pages 293–324. Elsevier, 2013.

[11] Yuan-Ming Lu and Ashvin Vishwanath. Theory and classification of interacting integer topological phases in two dimensions: A chern-simons approach. *Phys. Rev. B*, 86:125119, Sep 2012.

[12] Scott D. Geraedts and Olexei I. Motrunich. Exact realization of integer and fractional quantum hall phases in models in. *Annals of Physics*, 334:288 – 315, 2013.

[13] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2 – 111, 2006. January Special Issue.

[14] Jennifer Cano, Meng Cheng, Michael Mulligan, Chetan Nayak, Eugeniu Plamadeala, and Jon Yard. Bulk-edge correspondence in (2 + 1)-dimensional abelian topological phases. *Phys. Rev. B*, 89:115116, Mar 2014.

[15] Ashvin Vishwanath and T. Senthil. Physics of three-dimensional bosonic topological insulators: Surface-deconfined criticality and quantized magnetoelectric effect. *Phys. Rev. X*, 3:011016, Feb 2013.

[16] Olexei I. Motrunich and Ashvin Vishwanath. Emergent photons and transitions in the o(3) sigma model with hedgehog suppression. *Phys. Rev. B*, 70(7):075104–, August 2004.

[17] T. Senthil, Ashvin Vishwanath, Leon Balents, Subir Sachdev, and Matthew P. A. Fisher. Deconfined quantum critical points. *Science*, 303:1490–1494, 2004.

[18] T. Senthil, Leon Balents, Subir Sachdev, Ashvin Vishwanath, and Matthew P. A. Fisher. Quantum criticality beyond the Landau–Ginzburg–Wilson paradigm. *Phys. Rev. B*, 70:144407, 2004.

[19] Chong Wang and T. Senthil. Boson topological insulators: A window into highly entangled quantum phases. *Phys. Rev. B*, 87:235122, Jun 2013.

[20] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen. Symmetry protected topological orders and the group cohomology of their symmetry group. *Phys. Rev. B*, 87:155114, Apr 2013.

[21] F. J. Burnell, Xie Chen, Lukasz Fidkowski, and Ashvin Vishwanath. Exactly soluble model of a three-dimensional symmetry-protected topological phase of bosons with surface topological order. *Phys. Rev. B*, 90:245122, Dec 2014.

[22] Xie Chen, Fiona J. Burnell, Ashvin Vishwanath, and Lukasz Fidkowski. Anomalous symmetry fractionalization and surface topological order.

[23] Parsa Bonderson, Chetan Nayak, and Xiao-Liang Qi. A time-reversal invariant topological phase at the surface of a 3d topological insulator. *Journal of Statistical Mechanics: Theory and Experiment*, 2013(09):P09016, 2013.

[24] Max A. Metlitski, C. L. Kane, and Matthew P. A. Fisher. Symmetry-respecting topologically ordered surface phase of three-dimensional electron topological insulators. *Phys. Rev. B*, 92:125111, Sep 2015.

[25] Chong Wang, Andrew C. Potter, and T. Senthil. Gapped symmetry preserving surface state for the electron topological insulator. *Phys. Rev. B*, 88:115137, Sep 2013.

[26] Chong Wang, Andrew C. Potter, and T. Senthil. Classification of interacting electronic topological insulators in three dimensions. *Science*, 343(6171):629–631, 2014.

[27] L. Fidkowski, X. Chen, and A. Vishwanath. Non-Abelian Topological Order on the Surface of a 3D Topological Superconductor from an Exactly Solved Model. *Phys. Rev. X*, 3(4):041016, October 2013.

[28] A. Kitaev. Talk at ipam conference. 2015.

[29] Chong Wang and T. Senthil. Interacting fermionic topological insulators/superconductors in three dimensions. *Phys. Rev. B*, 89:195124, May 2014.

[30] Max A. Metlitski, Lukasz Fidkowski, Xie Chen, and Ashvin Vishwanath. Interaction effects on 3d topological superconductors: surface topological order from vortex condensation, the 16 fold way and fermionic kramers doublets. 2014.

[31] Anton Kapustin, Ryan Thorngren, Alex Turzillo, and Zitao Wang. Fermionic symmetry protected topological phases and cobordisms. 2014.

[32] E. Witten. Fermion path integrals and topological phases. 2015.

[33] Max A. Metlitski, C. L. Kane, and Matthew P. A. Fisher. Bosonic topological insulator in three dimensions and the statistical witten effect. 2013.

[34] T. Senthil. Symmetry-protected topological phases of quantum matter. *Annual Review of Condensed Matter Physics*, 6(1):299–324, 2015.